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Wavefunctions for the Luttinger liquid

K.-V. Pham[a], M. Gabay, and P. Lederer

Laboratoire de Physique des Solides[b], Université Paris–Sud, 91405 Orsay Cedex, France

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Abstract. Standard bosonization techniques lead to phonon-like excitations in a Luttinger liquid (LL), reflecting the absence of Landau quasiparticles in these systems. Yet in addition to the above excitations some LL are known to possess solitonic states carrying fractional quantum numbers (*e.g.* the spin 1/2 Heisenberg chain). We have reconsidered the zero modes in the low-energy spectrum of the Gaussian boson LL Hamiltonian both for fermionic and bosonic LL: in the spinless case we find that two elementary excitations carrying fractional quantum numbers allow to generate all the charge and current excited states of the LL. We explicitly compute the wavefunctions of these two objects and show that one of them can be identified with the 1D version of the Laughlin quasiparticle introduced in the context of the Fractional Quantum Hall effect. For bosons, the other quasiparticle corresponds to a spinon excitation. The eigenfunctions of Wen's chiral LL Hamiltonian are also derived: they are quite simply the one dimensional restrictions of the 2D bulk Laughlin wavefunctions.

PACS. 71.10.Pm Fermions in reduced dimensions (anyons, composite fermions, Luttinger liquid, etc.) – 71.27.+a Strongly correlated electron systems; heavy fermions

Many one dimensional gapless quantum systems admit a low-energy effective description similar to that of the Tomonaga-Luttinger model. This led Haldane to propose the phenomenology of the Luttinger liquid (LL) to describe these models [1]. Luttinger liquids are quantum critical theories and in the language of conformal field theories (CFT) they are said to belong to the $c = 1$ universality class [2]. The Gaussian boson Hamiltonian is a member of that class, which helps explain why the LL phenomenology is in essence that of an harmonic acoustic Hamiltonian. An important property of the LL is the absence of Landau quasiparticles [3]: it is therefore often believed that the only relevant excitations in a LL are phonons. Yet let us consider the Heisenberg spin chain which is a well known LL [4]; from Bethe ansatz one finds indeed that $\Delta S = 1$ excitations are not magnons, in line with the expectation that there are no Landau quasiparticles in a LL. In fact $\Delta S = 1$ excitations consist of two spin one-half spinons forming a continuum [5]. It is important to stress that spinons are fractional excitations of a spin chain (*i.e.* carrying fractional quantum numbers) since ΔS must be an integer for physically allowed excitations: this is quite clear if we perform Jordan-Wigner or Holstein-Primakov transformations on the spin Hamiltonian because then a spin $\Delta S = 1/2$ excitation becomes a charge $Q = 1/2$ excitation in a model for fermions or bosons. Yet it is noteworthy that there exists no first principle derivation of fractional excitations within the framework of the LL which maps given models onto the LL Gaussian boson Hamiltonian. The description in the standard -non chiral- LL should be contrasted with that of Wen's chiral LL, a variant of the former LL, used to describe edge states for the Fractional Quantum Hall Effect (FQHE) [6] where besides phonons one has fractional charge excitations: the Laughlin quasiparticles. Actually a first step aiming at including other types of excitations besides phonons for non-chiral LL was taken by Haldane in his own rigorous solution of the Tomonaga-Luttinger model, which yielded neutral collective density modes (phonons) plus charge and current excitations [1]. The latter excitations are generated by the zero mode part of the boson Hamiltonian and are customary in CFT [2]. In the non-interacting case Landau quasiparticles created at $\pm k_F$ are exact eigenstates of these zero modes; however they are not elementary excitations any more when interactions are turned on.

Our paper revisits those charge and current excitations for the generic (non-chiral) LL boson Hamiltonian. Due to chiral separation these charge and current excitations split into two independent chiral components, a property familiar from CFT; we will show that each of these chiral components must be viewed as a *composite excitation* built from two elementary fractional objects. One of these elementary excitations will be shown to be a Laughlin quasiparticle [7]; this will be established by computing the ground state and excitations eigenfunctions for the LL boson Hamiltonian: we will prove that the ground state and one of the elementary excitations have wavefunctions which are the 1D analogs of FQHE Laughlin

[a] e-mail: pham@lps.u-psud.fr
[b] associé au CNRS

wavefunctions. The nature of the other elementary excitation depends on whether one considers fermions or bosons: for bosons (and spins), we recover a charge one-half object (the spinon) while for fermions we end up with a novel object coming from the decay of the electron. We will also derive the eigenfunctions for Wen's chiral LL for the case of filling fractions $\nu = 1/(2n+1)$.

The Gaussian boson Hamiltonian is just a sum of harmonic oscillators so that the determination of the ground state and of its excitations is easy:

$$H_{\mathrm{B}} = \frac{u}{2}\int_0^L \mathrm{d}x\ K^{-1}(\nabla\Phi)^2 + K(\nabla\Theta)^2 \qquad (1)$$

where Θ and $\Pi = \nabla\Phi$ are canonical conjugate boson fields, $j = \frac{1}{\sqrt{\pi}}\nabla\Theta$ and $\delta\rho = -\frac{1}{\sqrt{\pi}}\nabla\Phi$ (j and $\delta\rho$ are respectively the particle current density and the particle density operators) and where $\Psi_{\mathrm{B}} = \rho^{1/2}\exp(i\sqrt{\pi}\Theta)$ for bosons, and $\Psi_{\mathrm{F}} = \Psi_{\mathrm{B}}(\exp(ik_F r - i\sqrt{\pi}\Phi) + \exp(-ik_F r + i\sqrt{\pi}\Phi))$ for fermions (we have multiplied Ψ_{B} by a Jordan-Wigner phase). u and K are the usual LL parameters [1]. The Fourier-transform of H_{B} is:

$$H_{\mathrm{B}} = \frac{u}{2}\sum_{q\neq 0} K^{-1}\Pi_q\Pi_{-q} + Kq^2\Theta_q\Theta_{-q} + \frac{\pi u}{2L}\left(\frac{\widehat{Q}^2}{K} + K\widehat{J}^2\right) \qquad (2)$$

where q is quantized as $q_n = 2\pi n/L$, $\widehat{Q}$ and $\widehat{J}$ are the charge and the current number operators. In the ground state, H_{B} reduces to a sum of harmonic oscillators (since $Q = 0 = J$); therefore the ground state is just a Gaussian:

$$\Psi_0 = \exp\left(-\frac{1}{2K}\sum_{n\neq 0}\frac{1}{|q_n|}\Pi_n\Pi_{-n}\right) \qquad (3)$$

returning to the original variables through $\Pi_q = \sqrt{\pi/L}\rho_q = \sqrt{\pi/L}\sum_i \exp(iqr_i)$, we find that Ψ_0 is nothing but a Laughlin wavefunction!

$$\Psi_0(\{r_{1,..,}r_{N_0}\}) = \prod_{i<j}|z_{ij}|^{1/K}. \qquad (4)$$

(We have defined $z_i = \exp i2\pi r_i/L$ and $z_{ij} = z_i - z_j$.) This is the correct form if we consider bosons; for fermions, we undo the singular (Jordan-Wigner) gauge transformation converting fermions to hard-core bosons so that $\psi_0^F = \prod_{i<j}(z_{ij})\ |z_{ij}|^{1/K-1}\ \exp i\pi\frac{N-1}{L}\sum r_i$. This derivation of the ground state follows exactly the same lines as that for the bosonic Landau-Ginzburg theory for the FQHE [8]. We note that equation (4) is also the exact ground state of the Calogero-Sutherland model [9]; the square modulus of the ground state functional of the Thirring model was also shown to be similar [10]. These simple results give a formal justification to the heuristic connection between 2D Laughlin wavefunctions and conformal blocks of CFTs [11]. One class of excited states corresponds to neutral (phonon-like) modes for which $Q = 0$ and $J = 0$. We find that the wavefunctions are simply Hermitte polynomials:

$$\left|n_{q_1}, n_{q_2}, ..., n_{q_p}\right\rangle = \prod_{s=1}^{p} H_{n_{q_s}}\left(\frac{\sum_i z_i^{q_s}}{\sqrt{LK\,|q_s|}}\right)|\Psi_0\rangle \qquad (5)$$

We now focus on excitations with non-zero values for Q and/or J. They were first singled out by Haldane in his rigorous approach to the Tomonaga-Luttinger model, and are standard in CFT. Such excitations are obtained from the vertex operators (the primary operators of the CFT) [2,3]:

$$V_{Q,J}(x) =: \exp i\sqrt{\pi}\,(J\Phi(x) - Q\Theta(x)) : \qquad (6)$$

where $\left[\widehat{Q}, V_{Q,J}\right] = QV_{Q,J}$ and $\left[\widehat{J}, V_{Q,J}\right] = JV_{Q,J}$. It follows from the periodicity requirement for the fields Ψ_{B} and Ψ_{F} that for bosons J must be an even integer while for fermions $Q - J$ is even (both Q and J are integers) [1]. We define also $V_{Q,J}(k_n)$ which carries a momentum k_n: $V_{Q,J}(k_n) = \int \mathrm{d}x \exp -i(k_n - k_F J)x\ V_{Q,J}(x)$. In the non-interacting case ($K = 1$) $V_{Q,J}(k_n)$ simply describe Q Landau quasiparticles. (The bosonization formulas show indeed that the electron is $V_{1,\pm 1}$.) However when $K \neq 1$ Landau quasiparticles are not elementary excitations any more (as suggested by the absence of quasiparticle poles in the electron Green function [3]): the paradigm set by the Heisenberg spin chain suggests that $V_{Q,J}(k_n)|0>$ decays into fractional excitations, much as the magnon is seen to be replaced by two spinons. This fractionalization of the spectrum of charge and current excitations of a LL is a direct consequence of the chiral separation of the boson Hamiltonian: $H_{\mathrm{B}} = H_+ + H_-$, and we summarize some known results below [2,3]:

$$H_\varepsilon = \frac{u}{4}\sum_{n\neq 0} Kq_n^2(-\varepsilon K^{-1}\Phi_n + \Theta_n)^2 + \frac{\pi u}{LK}\widehat{Q}_\varepsilon^2 \qquad (7)$$

where $\widehat{Q}_\varepsilon = \left(\widehat{Q} + \varepsilon K\widehat{J}\right)/2$ are chiral charges; $\Phi_\pm = \Phi \mp K\Theta$ are *free* chiral fields: $\Phi_\pm(x,t) = \Phi_\pm(x \mp ut)$. It is easy to check that H_+ and H_- commute; that property is routinely used in CFT. We also introduce $\Theta_\pm = \Theta \mp \frac{\Phi}{K}$ which are free fields. In terms of $\Theta_\pm$ the charge and current excitation operators $V_{Q,J}$ read:

$$V_{Q,J}(x) = \exp -i\sqrt{\pi}Q_+\Theta_+ \exp -i\sqrt{\pi}Q_-\Theta_-. \qquad (8)$$

When one adds Q electrons (or bosons) to the system, one gets therefore two counter-propagating parts carrying respectively a charge Q_+ and a charge Q_-. We define chiral excitation operators:

$$W^{\pm}_{Q_\pm}(x) = \exp -i\sqrt{\pi}Q_\pm\Theta_\pm \qquad (9)$$

which create a charge $Q_\pm$ (indeed $\left[\widehat{Q}, \exp -i\sqrt{\pi}Q_\pm\Theta_\pm\right] = Q_\pm \exp -i\sqrt{\pi}Q_\pm\Theta_\pm$) and are easily shown to obey anyonic commutation relations with statistics: $\mp\pi Q_\pm^2/K$ [12]. The exponential in the Fourier transform $W^{\pm}_{Q_\pm}(k_n)$ can be expanded as the product of a zero mode part and of muti-phonon processes, which shows $W^{\pm}_{Q_\pm}(k_n)$ is an exact eigenstate of $H_\pm$ [12]. Such an analysis in terms of chiral separation which is standard for conformal field theorists implies that even for a spinless LL there is a fractionalization of the electron with in general non-integral charge excitations (since $Q_\pm$ is not in general an integer), a fact generally not sufficiently appreciated in the condensed matter community.

After that summary, we now establish novel results concerning the elementary excitations for a non-chiral LL[1]. For the above chiral excitations we would like to find a basis of elementary excitations, *i.e.* identify objects from which all the other excitations can be built. We must carefully distinguish between Bose and Fermi statistics because of the constraints on Q and J. Let us consider first bosons: since J is even we can rewrite it as $J = 2n$ where n is an arbitrary integer. But then for *bosons*:

$$(Q_+, Q_-) = \left(\frac{Q+KJ}{2}, \frac{Q-KJ}{2}\right) = Q\,\left(\frac{1}{2}, \frac{1}{2}\right) + n\,\left(K, -K\right). \tag{10}$$

For bosons a general excitation is therefore constructed by applying $\left(W^{\pm}_{1/2}\right)^Q \left(W^{\pm}_{\pm K}\right)^n$ to Ψ_0, where Q and n are *independent integers* of arbitrary sign, which means that $W^{\pm}_{1/2}$ and $W^{\pm}_{\pm K}$ are *elementary excitations*. (Going back to reciprocal space this means that the exact eigenstate $W^{\pm}_{Q_\pm}(k_0)$ is built from states $\prod_{i=1}^{Q} W^{\pm}_{1/2}(q_i)\ \prod_{j=1}^{n} W^{\pm}_{K}(\widetilde{q}_j)$ where $k_0 = \sum_i q_i + \sum_j \widetilde{q}_j$.) These *two* types of elementary excitations are generated in the following processes: (i) adding one particle into the LL but no current ($Q = 1, n = J/2 = 0$) results in two (chiral) charge 1/2 objects (with statistics $\pi/4K$ [12]), moving with opposite velocities u, namely $W^{\pm}_{1/2}$; in spin problems (spins are hard-core bosons) $W^{\pm}_{1/2}$ is naturally interpreted as a $S = 1/2$ spinon; (ii) creating current without addition of a particle ($Q = 0, n = 1$) results in charges K and $-K$ moving with opposite velocities u. To identify the nature of these objects we compute for instance $W^{+}_{K}(z)\Psi_0$ in first quantization; using equation (4) and $\exp -i\sqrt{\pi}Q\Theta(x) = \exp Q\frac{\delta}{\delta\rho(x)}$ gives:

$$\left[\exp Q\frac{\delta}{\delta\rho(x)}\right] \exp\frac{1}{2K}\int\int \rho(y)\ln\left|\sin\frac{\pi}{L}(y-y')\right|\rho(y') = \exp\frac{Q}{K}\int\rho(y)\ln\left|\sin\frac{\pi}{L}(y-x)\right|\mathrm{d}y\Psi_0 = C\prod_i |z_i - z|^{Q/K}\,\Psi_0 \tag{11}$$

(C is an unessential constant). Since $\exp i\sqrt{\pi}J\Phi(x) = \exp i\pi J\int_0^L \theta(x-y)\delta\rho(y)\mathrm{d}y$ where $\theta(x) = \frac{1}{i\pi}\ln[\frac{(-x)}{|x|}]$ is the step function, $\exp i\sqrt{\pi}J\Phi(x) = \prod_i\left[(z_i - z)\,/\,|z_i - z|\right]^J$ $\exp -iJk_{\mathrm{F}}(2x + \sum_i r_i/N)$. Thus

$$W^{+}_{K}(x)\Psi_0 = \prod_i (z_i - z)\prod_{i<j}|z_{ij}|^{1/K}\exp -ik_{\mathrm{F}}\left(2x + \frac{\sum_i r_i}{N}\right) \tag{12}$$

This leads us to identify W^{+}_{K} with a Laughlin quasiparticle; the charge deduced from a plasma analogy is of course K in agreement with the above considerations. A simple argument may allow to appreciate the full parallel between the way Laughlin quasiparticles are generated in the FQHE and in a (non-chiral) LL. Laughlin showed that insertion of a flux quantum creates a fractionally charged Laughlin quasihole in the bulk [7]; by charge conservation the opposite charge is created at the edge [13]. We now show that threading flux in a LL ring also leads to the creation of a Laughlin quasiparticle–quasihole pair. Insertion of flux ϕ leads to the replacement $\frac{\pi u}{2L}K\widehat{J}^2 \longrightarrow \frac{\pi u}{2L}K\left(\widehat{J} + 2\frac{\phi}{\phi_0}\right)^2$ in equation (2) [14]. When $\phi = \phi_0$ the energy is minimized for $J = -2$ and the state is $V_{0,-2} = \exp -i2\sqrt{\pi}\Phi$. This is a ($Q = 0, J = -2$) process which by the previous analysis corresponds to the creation of W^{+}_{K} and W^{-}_{-K} a Laughlin quasiparticle–quasihole pair. This is entirely analogous to Laughlin's thought experiment. We note that Fisher and Glazman had previously argued for the existence of charge K Laughlin quasiparticles in a LL, but their argument was heuristic, relying on a study of backscattering by a barrier in a LL [15], while the existence of charge K elementary excitations was derived in this paper from first principles. Since fractional charge does not mean Laughlin quasiparticle (charge 1/3 solitons were known from Su and Schrieffer's work on charge density waves [16]), it is also necessary to prove that these charge K objects actually are genuine Laughlin quasiparticles: this was established in our computation of the wavefunction for W^{+}_{K}.

Let us illustrate these results on the specific example of the anisotropic Heisenberg XXZ spin chain. It is crucial to choose the right selection rules for Q and J: either bosonic or fermionic; here the correct ones are the bosonic ones because $S = 1/2$ spins are bosons (spins are *not* Jordan-Wigner fermions due to the string factor). Then consider spin transitions: $\Delta S = 1$ (*i.e.* $Q = 1, J = 0$); when we vary the anisotropy we observe a continuum of excitations which can be identified with spinons. This is recovered easily: indeed for $Q = 1, J = 0$ the previous results show that we generate $W^{+}_{1/2}$ and $W^{-}_{1/2}$. (Note that at the isotropic point for which $K = 1/2$ our spin half $W^{+}_{1/2}$ has a semionic statistics $\pi/2$ but in general its exchange statistics is $\pi/4K$ [12]. The spinon operator and its exclusion statistics have been considered in the $SU(2)$ symmetric case – the isotropic point – in [17].) The spinon operator $W^{+}_{1/2}$ upon acting on the ground state yields

$$W^{\pm}_{1/2}(z)\Psi_0 = \prod_i (z_i - z)^{1/2K}\prod_{i<j}|z_{ij}|^{1/K}\mathrm{e}^{-\frac{ik_F}{2K}(2x+\sum_i r_i/N)}. \tag{13}$$

Besides spinons we have the novel result that there should also be spin K Laughlin quasiparticles generated by spin currents. Imposing a twist in the boundary conditions will give rise to a Laughlin quasiparticle–quasihole pair of charge $\pm K$. We note that for the special case $K = 1/2$ (the isotropic Heisenberg chain, or boson self-dual point) elementary excitations consist solely of spinons and antispinons.

[1] For notational convenience we will work in direct space as is customary in CFT, the relevant exact excitations being obtained by Fourier transform.

We now turn to fermions and we may write $Q-J=2n$. Then for *fermions*:

$$(Q_+, Q_-) = Q \left(\frac{1+K}{2}, \frac{1-K}{2}\right) - n\ (K, -K) \tag{14}$$

Once again we may take $(Q=1, n=0)$; this corresponds to chiral charges $\frac{1-K}{2}$ and $\frac{1+K}{2}$ moving at velocities $-u$ and u respectively. The total current is uK (in units of k_{F}): indeed $n=0$ implies $J=1$, highlighting the fact that for fermions elementary excitations may mix charge and current states. Another generator is obtained with $(Q=0, n=1)$: it is a pure current state made out of counterpropagating Laughlin quasiparticle $(+K)$ and anti-particle $(-K)$. Note that in the non-interacting case $(K=1)$ these quasiparticles reduce to bare particles populating states near $\pm k_{\mathrm{F}}$. The general excitation is again built by creating an integer number of times $W^{\pm}_{\frac{1\pm K}{2}}$ and/or $W^{\pm}_{\pm K}$ states which precisely means that we have identified a set of elementary excitations for the fermionic LL.

The elementary excitations we have derived form a basis from which all charge and current excitations are obtained; by no means are they the only choice of basis: other bases of elementary excitations are obtained by considering base changes matrices with integer entries whose inverses are also integer-valued, which ensures that all excitations are integral linear combinations of the elementary excitations. For instance for fermions another basis is $W^{\pm}_{\frac{1\pm K}{2}}$ and $W^{\pm}_{1}$:

$$(Q_+, Q_-) = J \left(\frac{1+K}{2}, \frac{1-K}{2}\right) + n\ (1,1) \tag{15}$$

it is actually a basis dual to the previous one (obtained under exchanges of $K \longleftrightarrow 1/K$ and $\Phi \longleftrightarrow \Theta$).

Turning to Wen's chiral LL for the filling fraction $\nu = 1/(2n+1)$ using Wen's modified expression for the electron operator $\Psi = \exp i(2n+1)\varphi$ and $H = 2\pi(2n+1)\sum_{k>0}\rho_k\rho_{-k}$ (with $\rho = \frac{1}{2\pi}\nabla\varphi$) one finds the ground state $\psi_0 = \prod_{i<j}(z_{ij})^{2n+1}\exp -ik_F\sum r_j$ which up to a current term is just the 1D restriction of Laughlin bulk wavefunction; similarly the fractional charge excitation $\exp i\varphi/(2n+1)$ yields the wavefunction $\prod_i(z_i - z)\prod_{i<j}(z_i - z_j)^{2n+1}$. To our knowledge this is the first time the wavefunctions of Wen's chiral LL are directly computed from the Hamiltonian: while a heuristic connection had already been made between Laughlin's bulk wavefunctions and 1D edge theories (2D bulk wavefunctions as conformal blocks of 1+1D CFTs), we believe our results make the relation quite transparent.

The previous discussion can be generalized to the LL with spin; one should consider multicomponent Laughlin wavefunctions $\psi_0(\{r_i, \sigma_i\}) = \prod_{i<j}|z_{ij}|^{g_{\sigma_i,\sigma_j}}$. The discussion of charge and current excitations in a spinful LL then parallels the spinless case; in the case of spin-charge separation one finds a basis of four elementary excitations among which a charge one holon and a spin half spinon, as well as "Laughlin holons" carrying a charge K_c and "Laughlin spinons" with spin $K_s/2$.

Experimental observation of the various fractional excitations considered in this paper is of course an important issue in the extent that there are several candidates for a realization of the LL: quasi-1D organics, quantum wires, gapless spin chains, quantum Hall edge states [3]; an intriguing possibility might also be chiral edges of compressible phases of the 2D electron gas in between Hall plateaux [18]. Relevant probes could be as in the FQHE shot noise experiments or resonant tunneling [19]. We summarize the novel results obtained in the present paper: we have identified the elementary excitations of the LL, which are fractional objects (carrying fractional charges); by computing their wavefunctions we have found that one of them is identified as a Laughlin quasiparticle and is generated by current excitations in a LL. This property is a natural consequence of the fact that the ground state of the LL boson Hamiltonian is nothing but a Laughlin wavefunction. We have also computed the eigenfunctions for Wen's chiral LL at filling fractions $\nu = 1/(2n+1)$ which are just the one dimensional restrictions of the 2D bulk states.

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